Provably Correct Implementation

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Correct implementation

Structural operational semantics is useful when **implementing language**.

A **correct implementation** of program consists of:

- the definition of **abstract machine**,
- the definition of the **meaning** of the abstract machine instructions by an operational semantics,
- the definition of **translation functions** that will map expressions and statements in the *Jane* language into sequences of such instructions.

The correctness result will then state that if we translate a program into code, and execute the code on the abstract machine, then we get the same result as we specified by the semantic functions $\mathscr{S}_{ns}$ and $\mathscr{S}_{os}$.

The **Definition of abstract machine** consists of:

- the **configuration** of abstract machine,
- the definition of **instructions** and their:
  - **syntax** definition, and
  - **operational semantics** of instructions.

Abstract machine configuration

The abstract machine AM has **configuration** of the form:

$$\langle c, st, s \rangle$$

where

- c is a **code**, i.e. the sequence of instructions to be executed,
- st is the **evaluation stack**, and
- s is the **storage**, expressed by state $s \in \mathbf{State}$ and is used to hold the values of variables.

We use the evaluation stack to evaluate arithmetic and Boolean expressions. Formally, **stacks** are enclosed into semantic domain

$$st \in \mathbf{Stack} = (\mathbf{Z} \cup \mathbf{B})^*$$

and each element is a stack – finite sequence of numbers and/or truth values, e.g.

$$v_1 : v_2 : t_1 : v_3 : \ldots : v_n$$

where v_i are numeric values and t_j are truth values, v_1 is on the top of stack.

Syntax of instructions

We shall define two syntactic domains:

- $ins \in \mathbf{Instr}$ – for **instructions**; and
- $c \in \mathbf{Code}$ – for **code**, sequences of instructions.

The instructions of **AM** are given by the abstract syntax:

$$\begin{aligned} ins ::= {} & \texttt{PUSH},n \mid \texttt{ADD} \mid \texttt{MULT} \mid \texttt{SUB} \mid \texttt{TRUE} \mid \texttt{FALSE} \\ & \mid \texttt{EQ} \mid \texttt{LE} \mid \texttt{AND} \mid \texttt{NEG} \mid \texttt{FETCH},x \mid \texttt{STORE},x \\ & \mid \texttt{EMPTYOP} \mid \texttt{BRANCH}(c, c) \mid \texttt{LOOP}(c, c), \end{aligned}$$

$$c ::= \varepsilon \mid ins : c,$$

where ε is an empty sequence.

Semantics of instructions

The semantics of the instructions of the abstract machine is given by an operational semantics. **Transition relation** specifies how to execute the instructions:

$$\langle c, st, s\rangle \Rrightarrow \langle c', st', s'\rangle.$$

Final configuration has a form

$$\langle \varepsilon, \varepsilon, s\rangle$$

which means that all instructions were executed on **AM** and the resulting state is s.

Semantics of instructions

Instruction $\texttt{PUSH}-n$ pushes a constant value n onto the stack.

$$\langle \texttt{PUSH}-n : c, st, s\rangle \Rrightarrow \langle c, \mathcal{N}[\![n]\!] : st, s\rangle \quad (1_{\mathrm{AM}})$$

Instructions `ADD`, `MULT` and `SUB` assume, that on the top of stack two numeric values $v_1, v_2 \in \mathbf{Z}$ are pushed. They are removed by instruction, arithmetic operation is applied and the result is pushed onto the stack.

$$\langle \texttt{ADD} : c, v_1 : v_2 : st, s\rangle \Rrightarrow \langle c, (v_1 \oplus v_2) : st, s\rangle \quad (2_{\mathrm{AM}})$$

$$\langle \texttt{MULT} : c, v_1 : v_2 : st, s\rangle \Rrightarrow \langle c, (v_1 \otimes v_2) : st, s\rangle \quad (3_{\mathrm{AM}})$$

$$\langle \texttt{SUB} : c, v_1 : v_2 : st, s\rangle \Rrightarrow \langle c, (v_1 \ominus v_2) : st, s\rangle \quad (4_{\mathrm{AM}})$$

Semantics of instructions

Instructions `TRUE` and `FALSE` push the constants $\mathbf{tt}$ and $\mathbf{ff}$, respectively, onto the stack :

$$\langle \texttt{TRUE} : c, st, s\rangle \quad \ggg \quad \langle c, \mathbf{tt} : st, s\rangle \ (5_{\mathrm{AM}})$$

$$\langle \texttt{FALSE} : c, st, s\rangle \quad \ggg \quad \langle c, \mathbf{ff} : st, s\rangle \ (6_{\mathrm{AM}})$$

Instruction `EQ` assumes two numeric values $v_1, v_2 \in \mathbf{Z}$ on the top of stack. They are removed from stack and the result of the relation $=$ (whether the values are equal) is pushed onto the stack:

$$\langle \texttt{EQ} : c, v_1 : v_2 : st, s\rangle \ggg \langle c, (v_1 = v_2) : st, s\rangle \ (7_{\mathrm{AM}})$$

Instruction `LE` assumes two numeric values $v_1, v_2 \in \mathbf{Z}$ on the top of stack. They are removed from stack and the result of the relation $\leq$ is pushed onto the stack:

$$\langle \texttt{LE} : c, v_1 : v_2 : st, s\rangle \ggg \langle c, (v_1 \leq v_2) : st, s\rangle \ (8_{\mathrm{AM}})$$

Semantics of instructions

Instruction `AND` assumes two Boolean values $t_1, t_2 \in \mathbf{B}$ on the top of stack. They are removed from stack and the result of conjunction $\wedge$ (value $\mathbf{tt}$ or $\mathbf{ff}$) is pushed onto the stack:

$$\langle \texttt{AND} : c, t_1 : t_2 : st, s\rangle \Longrightarrow\!\!\gg \begin{cases} \langle c, \mathbf{tt} : st, s\rangle, & \text{if } t_1 = \mathbf{tt} \text{ and } t_2 = \mathbf{tt}, \\ \langle c, \mathbf{ff} : st, s\rangle, & \text{if } t_1 = \mathbf{ff} \text{ or } t_2 = \mathbf{ff}. \end{cases} \quad (9_{\mathrm{AM}})$$

Instruction `NEG` assumes one Boolean value $t \in \mathbf{B}$ on the top of stack. This value is removed from the stack and their logical negation is pushed onto the stack:

$$\langle \texttt{NEG} : c, t : st, s\rangle \Longrightarrow\!\!\gg \begin{cases} \langle c, \mathbf{tt} : st, s\rangle, & \text{if } t = \mathbf{ff}; \\ \langle c, \mathbf{ff} : st, s\rangle, & \text{if } t = \mathbf{tt}. \end{cases} \quad (10_{\mathrm{AM}})$$

Semantics of instructions

Instruction $\texttt{FETCH}-x$ pushes the value bound to x onto the stack - a value in actual state $s\ x$:

$$\langle \texttt{FETCH}-x : c, st, s\rangle \Rrightarrow \langle c, (s\ x) : st, s\rangle \quad (11_{\mathrm{AM}})$$

Instruction $\texttt{STORE}-x$ pops the topmost element off the stack and updates the storage so that the popped value is bound to x:

$$\langle \texttt{STORE}-x : c, v : st, s\rangle \Rrightarrow \langle c, st, s[x \mapsto v]\rangle \quad (12_{\mathrm{AM}})$$

Instructions $\texttt{EMPTYOP}$ is an empty instruction (operation), it changes neither the state nor the stack:

$$\langle \texttt{EMPTYOP} : c, st, s\rangle \Rrightarrow \langle c, st, s\rangle \quad (13_{\mathrm{AM}})$$

Semantics of instructions

Instruction $\texttt{BRANCH}(c_1, c_2)$ changes the flow control.

If the top of the stack is the value $\mathbf{tt}$ (that is some Boolean expression has been evaluated to true) then the stack is popped and c_1 is to be executed in the next step. Otherwise, if the top element of the stack is $\mathbf{ff}$ then it will be popped and c_2 will be executed:

$$\langle \texttt{BRANCH}(c_1, c_2) : c, t : st, s\rangle \Longrightarrow \begin{cases} \langle c_1 : c, st, s\rangle, & \text{if } t = \mathbf{tt}, \\ \langle c_2 : c, st, s\rangle, & \text{if } t = \mathbf{ff}. \end{cases} \quad (14_{\mathrm{AM}})$$

Semantics of instructions

A looping construct such as the `while`-construct can be implemented using the instruction $\texttt{LOOP}(c_1, c_2)$. The semantics of this instruction is defined by rewriting it to a combination of other constructs including the `BRANCH`-instruction and itself:

$$\langle \texttt{LOOP}(c_1, c_2) : c, st, s \rangle$$

$$\Longrightarrow\!\!\!\!\Longrightarrow \langle c_1 : \texttt{BRANCH}(c_2 : \texttt{LOOP}(c_1, c_2), \texttt{EMPTYOP}) : c, st, s \rangle \qquad (15_{\mathrm{AM}})$$

We emphasize that the assumptions for every instruction are necessary for the continuation of **AM** work.

If the **AM** is expecting value(s) on the top of stack and

- on the top of stack the values are missing, or
- the type of values does not correspond

then the execution of abstract machine **stops**.

Computation sequence

The **execution of program** on **AM** is expressed by computation sequence. Given a sequence c of instructions and a storage s, a **computation sequence** for c and s is either:

- a **finite sequence** of configurations

  $$\alpha_0, \alpha_1, \ldots, \alpha_n$$

  satisfying

  $$\alpha_0 = \langle c, \varepsilon, s \rangle, \qquad \alpha_i \Rrightarrow \alpha_{i+1}, \qquad \alpha_n = \langle \varepsilon, st, s' \rangle,$$

  for $0 \leq i < k$, $k \geq 0$ and where there is no α such that $\alpha_n \Rrightarrow \alpha$. Such sequence **terminates**.
- a **finite sequence** of configurations as above but

  $$\alpha_n = \langle c, st, s \rangle.$$

  Here the execution of program is **stopped**.
- an **infinite sequence** of configurations

  $$\alpha_0, \alpha_1, \ldots$$

  We say that the sequence is looping – it is infinite.

Example

We consider the following program for abstract machine:

$$\texttt{PUSH}-1 : \texttt{FETCH}-x : \texttt{ADD} : \texttt{STORE}-x$$

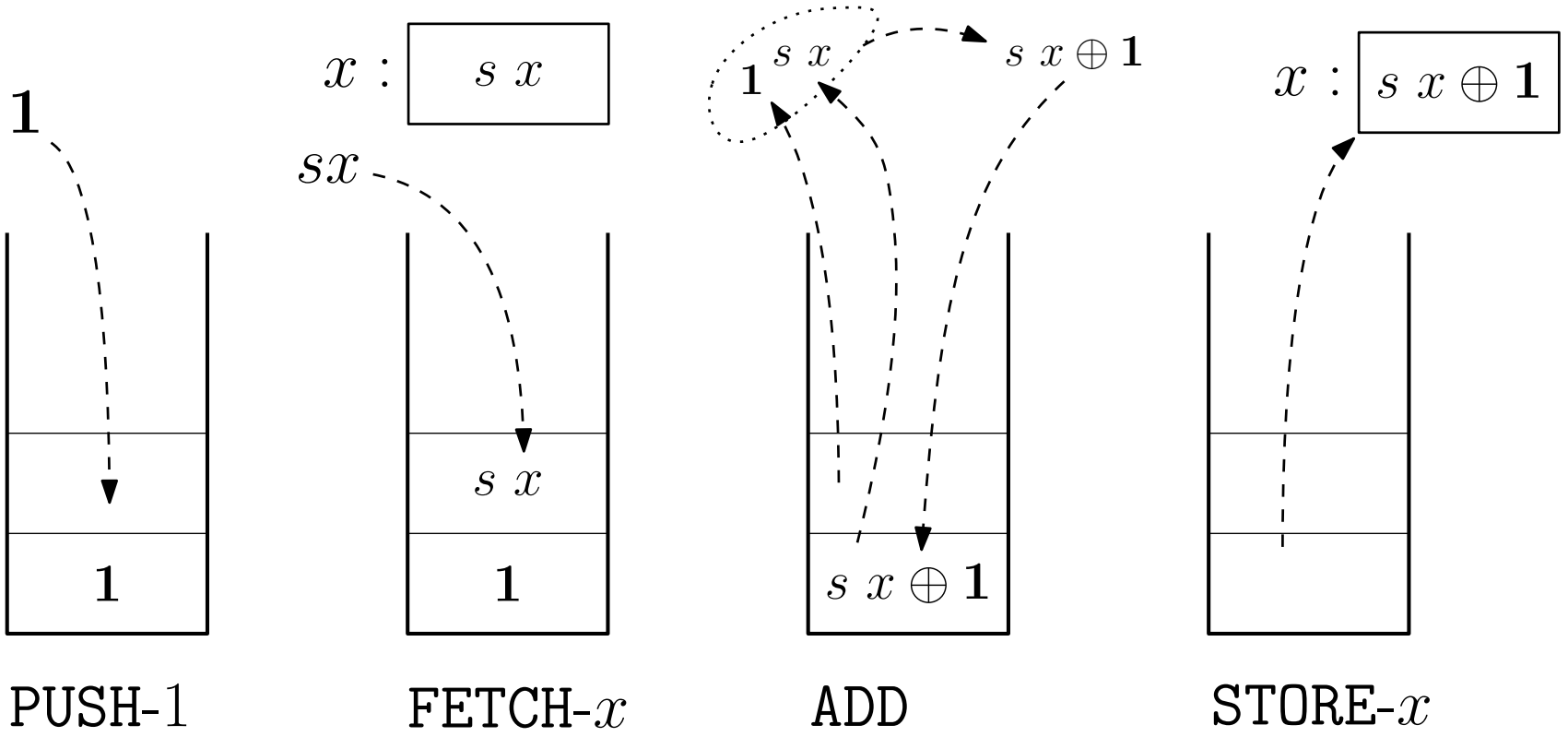
with an initial state $s\ x = \mathbf{5}$.

A computation sequence of this program is:

$$\begin{aligned}
\alpha_0 &= \langle \texttt{PUSH}-1 : \texttt{FETCH}-x : \texttt{ADD} : \texttt{STORE}-x, \varepsilon, s \rangle \ \ggg \\
\alpha_1 &= \langle \texttt{FETCH}-x : \texttt{ADD} : \texttt{STORE}-x, \mathbf{1}, s \rangle \ \ggg \\
\alpha_2 &= \langle \texttt{ADD} : \texttt{STORE}-x, \mathbf{5} : \mathbf{1}, s \rangle \ \ggg \\
\alpha_3 &= \langle \texttt{STORE}-x, \mathbf{6}, s \rangle \ \ggg \\
\alpha_4 &= \langle \varepsilon, \varepsilon, s[x \mapsto \mathbf{6}] \rangle
\end{aligned}$$

Computation sequence terminates and the resulting state is $s\ x = \mathbf{6}$.

Example



□

Properties of computation sequence

We denote

- $\langle c, st, s\rangle \Rrightarrow^{k} \langle c', st', s'\rangle$ – computation sequence of length k,
- $\langle c, st, s\rangle \Rrightarrow^{*} \langle c', st', s'\rangle$ – finite computation sequence.

Lemma 1: Let $c_1, c_2, c' \in \mathbf{Code}$ be codes, $st_1, st_2, st' \in \mathbf{Stack}$ are stacks and $s, s' \in \mathbf{State}$ are states of **AM**. If

$$\langle c_1, st_1, s\rangle \Rrightarrow^{k} \langle c', st', s'\rangle$$

then

$$\langle c_1 : c_2, st_1 : st_2, s\rangle \Rrightarrow^{k} \langle c' : c_2, st' : st_2, s'\rangle.$$

□

This means that we can extend the code component as well as the stack component without changing the behavior of the machine.

Properties of computation sequence

Lemma 2: Let $c_1, c_2 \in \mathbf{Code}$ be codes, $st, st', st'' \in \mathbf{Stack}$ are stack and $s, s', s'' \in \mathbf{State}$ are states of **AM**. If for some natural k holds

$$\langle c_1 : c_2, st, s\rangle \Longrightarrow^{k} \langle \varepsilon, st'', s''\rangle$$

then exists configuration $\langle \varepsilon, st', s'\rangle$ and natural numbers k_1, k_2 with $k_1 + k_2 = k$ such that

$$\langle c_1, st, s\rangle \Longrightarrow^{k_1} \langle \varepsilon, st', s'\rangle \quad \text{and}$$

$$\langle c_2, st', s'\rangle \Longrightarrow^{k_2} \langle \varepsilon, st'', s''\rangle.$$

□

This means that the execution of a composite sequence of instructions can be split into two sub-sequences.

Properties of computation sequence

Lemma 3: Semantics of abstract machine is **deterministic** if for all elements $\alpha, \alpha', \alpha''$:

$$\text{if} \quad \alpha \ggg \alpha' \quad \text{and} \quad \alpha \ggg \alpha'', \text{ then} \quad \alpha' = \alpha''.$$

□

We shall define the **meaning** of a sequence of instructions as a (partial) function from $\mathbf{State}$ to $\mathbf{State}$:

$$\mathscr{M}[\![c]\!]s = \begin{cases} s', & \text{if} \quad \langle c, \varepsilon, s\rangle \ggg^* \langle \varepsilon, st, s'\rangle, \\ \\ \bot, & \text{otherwise.} \end{cases}$$

Specification of the translation

A **translation** of *Jane* into instructions of **AM** – a **code generating** is defined:

- by **translation functions**,
- for each syntactic domain we define **one** translation function,
- and we define it for **all** alternatives in the given production rule.

Specification of the translation

Translation of **arithmetic expressions** is defined by function

$$\mathscr{TE} : \mathbf{Expr} \to \mathbf{Code}$$

for all syntactic forms of arithmetic expressions in *Jane*:

$$\mathscr{TE}[\![n]\!] = \texttt{PUSH}{-}n,$$

$$\mathscr{TE}[\![x]\!] = \texttt{FETCH}{-}x,$$

$$\mathscr{TE}[\![e_1 + e_2]\!] = \mathscr{TE}[\![e_2]\!] : \mathscr{TE}[\![e_1]\!] : \texttt{ADD},$$

$$\mathscr{TE}[\![e_1 * e_2]\!] = \mathscr{TE}[\![e_2]\!] : \mathscr{TE}[\![e_1]\!] : \texttt{MULT},$$

$$\mathscr{TE}[\![e_1 - e_2]\!] = \mathscr{TE}[\![e_2]\!] : \mathscr{TE}[\![e_1]\!] : \texttt{SUB}.$$

Note: Code generated for binary expressions consists of the code for the *right* argument followed by that for the *left* argument and finally the appropriate instruction for the operator.

Specification of the translation

Translation of **Boolean expressions** is defined by function:

$$\mathscr{TB} : \mathbf{Bexpr} \to \mathbf{Code}$$

for all syntactic forms of Boolean expressions in *Jane*:

$$\mathscr{TB}[\![\texttt{true}]\!] = \texttt{TRUE},$$

$$\mathscr{TB}[\![\texttt{false}]\!] = \texttt{FALSE},$$

$$\mathscr{TB}[\![e_1 = e_2]\!] = \mathscr{TE}[\![e_2]\!] : \mathscr{TE}[\![e_1]\!] : \texttt{EQ},$$

$$\mathscr{TB}[\![e_1 \leq e_2]\!] = \mathscr{TE}[\![e_2]\!] : \mathscr{TE}[\![e_1]\!] : \texttt{LE},$$

$$\mathscr{TB}[\![\neg b]\!] = \mathscr{TB}[\![b]\!] : \texttt{NEG},$$

$$\mathscr{TB}[\![b_1 \wedge b_2]\!] = \mathscr{TB}[\![b_2]\!] : \mathscr{TB}[\![b_1]\!] : \texttt{AND}.$$

Example 1

Consider an arithmetic expression $x * (x - 1)$ and generate a code for **AM**:

$$\mathscr{TE}[\![x * (x-1)]\!] = \mathscr{TE}[\![x-1]\!] : \mathscr{TE}[\![x]\!] : \texttt{MULT}$$

$$= \mathscr{TE}[\![x-1]\!] : \texttt{FETCH}-x : \texttt{MULT}$$

$$= \mathscr{TE}[\![1]\!] : \mathscr{TE}[\![x]\!] : \texttt{SUB} : \texttt{FETCH}-x : \texttt{MULT}$$

$$= \texttt{PUSH}-1 : \texttt{FETCH}-x : \texttt{SUB} : \texttt{FETCH}-x : \texttt{MULT}$$

□

Specification of the translation

Translation of **statements** is defined by the function:

$$\mathscr{TS} : \mathbf{Statm} \to \mathbf{Code}$$

for all syntactic forms of statements in *Jane*:

$$\mathscr{TS}[\![x := e]\!] = \mathscr{TE}[\![e]\!] : \texttt{STORE}-x,$$

$$\mathscr{TS}[\![\texttt{skip}]\!] = \texttt{EMPTYOP},$$

$$\mathscr{TS}[\![S_1; S_2]\!] = \mathscr{TS}[\![S_1]\!] : \mathscr{TS}[\![S_2]\!],$$

$$\mathscr{TS}[\![\texttt{if } b \texttt{ then } S_1 \texttt{ else } S_2]\!] = \mathscr{TB}[\![b]\!] : \texttt{BRANCH}(\mathscr{TS}[\![S_1]\!], \mathscr{TS}[\![S_2]\!]),$$

$$\mathscr{TS}[\![\texttt{while } b \texttt{ do } S]\!] = \texttt{LOOP}(\mathscr{TB}[\![b]\!], \mathscr{TS}[\![S]\!]).$$

Example 2

Consider a simple program $z{:=}0;\,\texttt{while}\ (y \le x)\ \texttt{do}\ (z{:=}z+1;\,x{:=}x-y)$ and generate a code for **AM**:

$$\mathscr{TS}[\![z{:=}0;\,\texttt{while}\ (y \le x)\ \texttt{do}\ (z{:=}z+1;\,x{:=}x-y)]\!]$$

$$= \mathscr{TS}[\![z{:=}0]\!] : \mathscr{TS}[\![\texttt{while}\ (y \le x)\ \texttt{do}\ (z{:=}z+1;\,x{:=}x-y)]\!]$$

$$= \mathscr{TE}[\![0]\!] : \texttt{STORE}{-}z : \texttt{LOOP}(\mathscr{TB}[\![y \le x]\!], \mathscr{TS}[\![z{:=}z+1;\,x{:=}x-y]\!])$$

$$= \texttt{PUSH}{-}0 : \texttt{STORE}{-}z \\ \quad : \texttt{LOOP}(\mathscr{TE}[\![x]\!] : \mathscr{TE}[\![y]\!] : \texttt{LE}, \mathscr{TS}[\![z{:=}z+1]\!] : \mathscr{TS}[\![x{:=}x-y]\!])$$

$$= \texttt{PUSH}{-}0 : \texttt{STORE}{-}z \\ \quad : \texttt{LOOP}(\texttt{FETCH}{-}x : \texttt{FETCH}{-}y : \texttt{LE}, \\ \quad\quad \texttt{PUSH}{-}1 : \texttt{FETCH}{-}z : \texttt{ADD} : \texttt{STORE}{-}z \\ \quad\quad : \texttt{FETCH}{-}y : \texttt{FETCH}{-}x : \texttt{SUB} : \texttt{STORE}{-}x)$$

□

The semantic function $\mathscr{S}_{AM}$

An **abstract implementation** of statement S is obtained by executing of the following steps:

- translating the statement S into code of **AM** by translation functions, and
- executing the code on **AM** by semantics of instructions.

This can be expressed by the **semantic function** $\mathscr{S}_{AM}$ of abstract machine:

$$\mathscr{S}_{AM} : \mathbf{Statm} \to (\mathbf{State} \rightharpoonup \mathbf{State})$$

which is defined for any statement $S \in \mathbf{Statm}$ as follows:

$$\mathscr{S}_{AM}[\![S]\!] = (\mathscr{TS} \circ \mathscr{M})[\![S]\!].$$

Correctness of abstract implementation

The **correctness of abstract implementation** amounts to showing that, if we first translate a statement into code for **AM** and then execute code, then we must to obtain the same result as specified by the operational semantics of *Jane*.

Proof will be done in three steps:

- for arithmetic expressions,
- for Boolean expressions,
- for statements.

Correctness of the implementation of arithmetic expressions

Theorem 1: For all arithmetic expressions e we have:

$$\langle \mathscr{TE}[\![e]\!], \varepsilon, s\rangle \Longrightarrow\!\!\gg^* \langle \varepsilon, \mathscr{E}[\![e]\!]s, s\rangle$$

Proof: The proof is by structural induction on e.

1. **The case** n:
   From translation function we have $\mathscr{TE}[\![n]\!] = \texttt{PUSH}-n$.
   From semantics of instruction it implies $\langle \texttt{PUSH}-n, \varepsilon, s\rangle \Longrightarrow\!\!\gg \langle \varepsilon, \mathscr{N}[\![n]\!], s\rangle$.
   Since

$$\mathscr{E}[\![n]\!]s = \mathscr{N}[\![n]\!]$$

   we have completed the proof in this case.

Correctness of the implementation of arithmetic expressions

2. **The case** x (variable):
   From translation function we have $\mathscr{TE}[\![x]\!] = \texttt{FETCH}-x$.
   From semantics of instruction it implies $\langle \texttt{FETCH}-x, \varepsilon, s\rangle \Rrightarrow \langle \varepsilon, (s\ x), s\rangle$.
   Since:

$$\mathscr{E}[\![x]\!]s = s\ x$$

   this is required result.

Correctness of the implementation of arithmetic expressions

3. **The case** $e_1 + e_2$:

   We have $\mathscr{TE}[\![e_1 + e_2]\!] = \mathscr{TE}[\![e_2]\!] : \mathscr{TE}[\![e_1]\!] : \texttt{ADD}$.

   The **induction hypothesis** (IH) applied to both expressions e_1, e_2 gives that:

$$\langle \mathscr{TE}[\![e_1]\!], \varepsilon, s\rangle \Rrightarrow^* \langle \varepsilon, \mathscr{E}[\![e_1]\!]s, s\rangle$$
$$\langle \mathscr{TE}[\![e_2]\!], \varepsilon, s\rangle \Rrightarrow^* \langle \varepsilon, \mathscr{E}[\![e_2]\!]s, s\rangle$$

   In both cases all intermediate configurations will have a non-empty evaluation stack. Thus:

$$\begin{array}{ll}
\langle \mathscr{TE}[\![e_2]\!] : \mathscr{TE}[\![e_1]\!] : \texttt{ADD}, \varepsilon, s\rangle \Rrightarrow^* & \text{from IH and Lemma 1} \\
\langle \mathscr{TE}[\![e_1]\!] : \texttt{ADD}, \mathscr{E}[\![e_2]\!]s, s\rangle \Rrightarrow^* & \text{from IH and Lemma 1} \\
\langle \texttt{ADD}, \mathscr{E}[\![e_1]\!]s : \mathscr{E}[\![e_2]\!]s, s\rangle & \text{from sem. AM} \\
\langle \varepsilon, \mathscr{E}[\![e_1]\!]s \oplus \mathscr{E}[\![e_2]\!]s, s\rangle &
\end{array}$$

   Since

$$\mathscr{E}[\![e_1 + e_2]\!]s = \mathscr{E}[\![e_1]\!]s \oplus \mathscr{E}[\![e_2]\!]s$$

   we have the desired result.

The proof for other cases is analogous.

Correctness of the implementation of Boolean expressions

Theorem 2: For all Boolean expressions b we have:

$$\langle \mathscr{T}\mathscr{B}[\![b]\!], \varepsilon, s\rangle \ggg^* \langle \varepsilon, \mathscr{B}[\![b]\!], s\rangle$$

Proof: The proof is analogous to the proof for correctness of arithmetic expressions.

□

Theorem 3: For every statement S of *Jane* we have:

$$\mathscr{S}_{ns}[\![S]\!] = \mathscr{S}_{AM}[\![S]\!]$$

Proof: The theorem is proved in two stages:

1. if $\langle S, s\rangle \to s'$ then $\langle \mathscr{T}\mathscr{S}[\![S]\!], \varepsilon, s\rangle \ggg^* \langle \varepsilon, \varepsilon, s'\rangle$;
2. if $\langle \mathscr{T}\mathscr{S}[\![S]\!], \varepsilon, s\rangle \ggg^k \langle \varepsilon, st, s'\rangle$ then $\langle S, s\rangle \to s'$ and $st = \varepsilon$.