while $(x<5)$ do $x := x+1$

$$\frac{\langle x:=x+1, s\rangle \to s' \quad \langle \text{while } (x<5) \text{ do } x:=x+1, s'\rangle \to s'' \quad \mathcal{B}[\![\ldots]\!]s = \underline{tt}}{\langle \text{while } (x<5) \text{ do } x:=x+1, s\rangle \to s''}$$

$\mathcal{B}[\![x<5]\!]s = \underline{tt}$

$\underline{\underline{s\,x<5}}$ it always terminates

$$\frac{\mathcal{B}[\![x<5]\!]s = \underline{ff}}{\langle \ldots\ldots \quad , s\rangle \to s}$$ $s\,x \geq \underline{5}$

terminates $\mathcal{B}[\![\neg(x=10)]\!]s = \underline{tt}$

$= \underline{ff}$

$s\,x = \underline{10}$

while $\neg(x=10)$ do $x := x*2$

$a_1 = s\,x$

$q = 2$

$a_n = a_1 \cdot q^{n-1}$

$10 = a_1\, 2^{n-1}$

$a_1 = \frac{1}{10}\, 2^{n-1}$

if false then while true do S' else skip

$f(s) = s[x \mapsto \underline{1}]$

$g(s) = s[x \mapsto \underline{2}]$

$f \neq g$ if $f\,s = s'$ then $g\,s = s'$

$s_0 = [x \mapsto \underline{7}]$ $f\,s_0 = s_0'$ $s_0' = s_0[x \mapsto \underline{1}]$

$g\,s_0 = s_0''$ $s_0'' = s_0[x \mapsto \underline{2}]$

$f_1\, n = n$

$f_2\, n = \begin{cases} n & n \geq 0 \\ \perp & \text{oth.} \end{cases}$

$f_3\, n = \begin{cases} n & n = 0 \\ \perp & \text{oth.} \end{cases}$

$f_4\, n = \begin{cases} n & n \leq 0 \\ \perp & \text{oth.} \end{cases}$

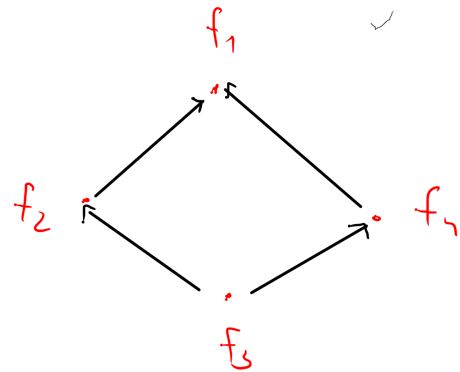
$f_1 \sqsubseteq f_1$ $\quad f_2 \sqsubseteq f_2$ $\quad f_3 \sqsubseteq f_3$

$f_2 \sqsubseteq f_1$ $\quad f_3 \sqsubseteq f_2$ $\quad f_3 \sqsubseteq f_4$

$f_3 \sqsubseteq f_1$ $\quad f_4 \not\sqsubseteq f_2$ $\quad f_4 \sqsubseteq f_4$

$f_4 \sqsubseteq f_1$

Hasse diagram



$n ::= 0 \mid \dots\ 7 \mid n0 \mid \dots \mid n7$

$\mathcal{N} : \underline{Oct} \to \mathbb{Z}$

$\mathcal{N}[\![0]\!] = \underline{0}$ $\qquad \mathcal{N}[\![n\,0]\!] = \underline{8} \otimes \mathcal{N}[\![n]\!]$

$\vdots$ $\qquad \vdots$

$\mathcal{N}[\![7]\!] = \underline{7}$ $\qquad \mathcal{N}[\![n\,7]\!] = \underline{8} \otimes \mathcal{N}[\![n]\!] \oplus \underline{7}$

DSL Robot

left 0 $\equiv$ skip

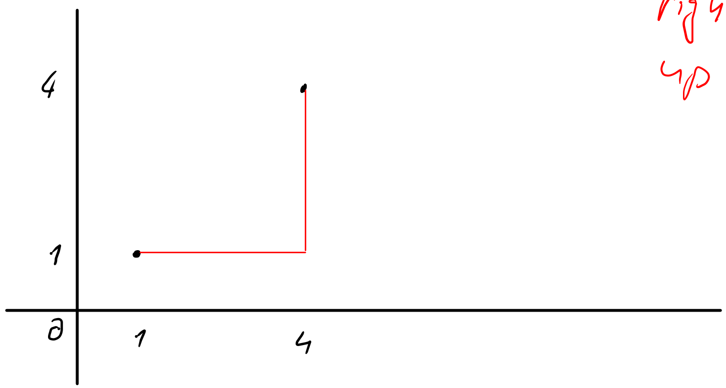
$\llbracket$ right ; up ; right ; up ; right ; up $\rrbracket (1,1)$

$= \llbracket$ up ; —"— $\rrbracket (2,1)$

$= \llbracket$ right ; —"— $\rrbracket (2,2)$

right 3 ;

up 3 ;



Denotational semantics

$z:=0;\ \text{while } (y \le x) \text{ do } (z:=z+1;\ x:=x-y)$

$S_1 \; ; \; S_2$

$$\mathcal{S}_{ds}[\![S_1; S_2]\!]s = \mathcal{S}_{ds}[\![S_2]\!](\mathcal{S}_{ds}[\![S_1]\!]s) = (\text{fix } F)\, s[z \mapsto \underline{0}]$$

$$(Fg)s = \text{cond}(\mathcal{B}[\![y \le x]\!], g \circ \mathcal{S}_{ds}[\![z:=z+1;\ x:=x-y]\!], \mathcal{S}_{ds}[\![\text{skip}]\!])\, s$$

$$= \text{cond}(\mathcal{B}[\![y \le x]\!], g(s[z \mapsto sz+1][x \mapsto sx-sy]), \text{id})\, s$$

$$(Fg)s = \begin{cases} s & sy > sx \\ g(s[z \mapsto sz+1][x \mapsto sx-sy]) & sy \le sx \end{cases}$$

$(F^0 \bot)s = \bot$

x	y
17	5
12	5
7	5
2	5

$$(F^1 \bot)s = \begin{cases} s & sy > sx \\ F(F^0 \bot s) = \bot & \text{otherwise} \end{cases}$$

$sy \le sx$

$$(F^2 \bot)s = \begin{cases} s & sy > sx \\ F(F^1 \bot s) = \begin{cases} s[z \mapsto sz+1][x \mapsto sx-sy] & sy > sx - sy,\ 2*sy > sx \\ \bot \end{cases} \end{cases}$$

$$(F^3 \bot)s = \begin{cases} s & sy > sx \\ F(F^2 \bot s) = \begin{cases} s[z \mapsto sz+1][x \mapsto sx-sy] & 2*sy > sx \\ F(F^1 \bot s) = \begin{cases} s[z \mapsto sz+2][x \mapsto sx - 2*sy] & sy > sx - 2*sy,\ 3*sy > sx \\ F^0 \bot \ldots = \bot \end{cases} \end{cases} \end{cases}$$

$$(F^n \bot)s = \begin{cases} s & sy > sx \\ s[z \mapsto sz+(n-1)][x \mapsto sx-(n-1)*sy] & n*sy > sx \\ \bot & \text{oth.} \end{cases}$$

$$(\text{fix } F)s = \begin{cases} s & sy > sx \\ s[z \mapsto sz + \overset{sx \text{ div } sy}{\lfloor \frac{sx}{sy} \rfloor}][x \mapsto \overset{sx \text{ mod } sy}{sx - \lfloor \frac{sx}{sy} \rfloor * sy}] & sy \le sx \\ \bot & sy = \underline{0} \end{cases}$$

CATEGORY THEORY

$Ob(\mathcal{C}) = \{A, B, C, \dots\}$

$Morph(\mathcal{C}) = \{f : A \to B,\ g : B \to C, \dots\}$

$Ar(\mathcal{C})$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "id_A"] \arrow[dr] & A \arrow[d, "f"] \arrow[dr] & \\
& B \arrow[r, "id_B"'] & B
\end{tikzcd}$$

$f \circ id_A =$

$= id_B \circ f$

FUNCTOR

$F: \mathcal{C} \to \mathcal{D}$

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "f"'] & F(A) \arrow[d, "F(f)"] \\
B \arrow[r, "F"'] & F(B)
\end{tikzcd}$$

$F(A) = FA$

$F(f) \circ F = F \circ f$

a memory state $s : \underline{Var} \to \mathbb{Z}$

$\underline{State} = \underline{Var} \to \mathbb{Z}$

a signature $\Sigma = (T, F)$

$T = \{\sigma_1, \dots \sigma_n\}$

$F = \{f : \sigma_1, \sigma_2 \to \sigma, \dots\}$

$pop(push(s, x)) = s$ (eq. logic)

(A, a) $a : T(A) \to A$

(C, φ) $\varphi : C \to Q(C)$

COALGEBRAIC SEMANTICS

$$\mathrm{config} = ([\![D^*; S^*]\!] m_0, \pi^*, \sigma^*)$$