Configuration for OS

$$\langle S, s \rangle \Rightarrow s'$$

Configuration for AM

$$\langle c, st, s \rangle \Rightarrow \langle c', st', s' \rangle$$

A transition closure (Kleene's closure, Kleene's star)

$A = \{1, 2, 3\}$ $\quad A^* = \{\varepsilon, 1, 2, 3, 11, 12, 13, 21, 22, 23, \ldots$

$L \subseteq A^*$ $\quad L(G) \subseteq A^*$

$st_1 = (1:2)$

$st' = \varepsilon$ empty stack

LOOP instruction

$\langle \mathrm{LOOP}\ (c_1, c_2) : c,\ st,\ s \rangle \Rightarrow\rangle$

$\Rightarrow\rangle \langle\ c_1 : \mathrm{BRANCH}(c_2 ; \mathrm{LOOP}\ (c_1, c_2),\ \mathrm{EMPTYOP}),\ st,\ s \rangle$

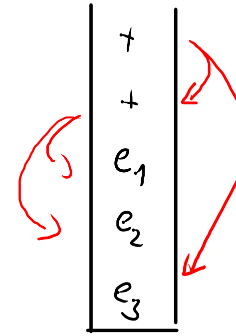
$\langle \text{while } b \text{ do } S, s \rangle \Rightarrow \langle \text{if } b \text{ then } (S; \text{ while } b \text{ do } S) \text{ else skip}), s \rangle$

$\mathcal{TB}[\![b]\!] = c_1$

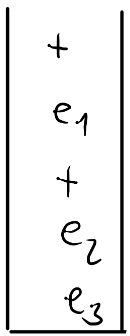
$\mathcal{TS}[\![S]\!] = c_2$

$(e_1+e_2)+e_3$ $\qquad\qquad$ $e_1+(e_2+e_3)$

$$TE[\![(e_1+e_2)+e_3]\!] = TE[\![e_3]\!] : TE[\![e_1+e_2]\!] : ADD =$$
$$= TE[\![e_3]\!] : TE[\![e_2]\!] : TE[\![e_1]\!] : ADD : ADD$$



$$TE[\![e_1+(e_2+e_3)]\!] = TE[\![e_2+e_3]\!] : TE[\![e_1]\!] : ADD =$$
$$= TE[\![e_3]\!] : TE[\![e_2]\!] : ADD : TE[\![e_1]\!] : ADD$$



mid := x + y + z - max - min;

ALTERNATIVE DEFINITION

$$\langle c, st, m\rangle \Rightarrow \langle c', st', m'\rangle$$

$ins ::= \dots \mid GET\text{-}n \mid PUT\text{-}n$

$m \in \underline{Memory}$ $\qquad$ $\underline{Memory} = \mathbb{Z}^*$ list of values

$$\mathcal{T}\mathcal{S}'[\![x := e]\!] = \mathcal{T}\mathcal{E}'[\![e]\!] : PUT\text{-}n$$

$$\mathcal{T}\mathcal{E}'[\![x]\!] = GET\text{-}n$$

$$\langle GET\text{-}n : c, st, m\rangle \Rightarrow\!\!\!\!\!\!\Rightarrow \langle c, m[i] : st, m\rangle$$

$$\langle PUT\text{-}n : c, v : st, m\rangle \Rightarrow\!\!\!\!\!\!\Rightarrow \langle c, st, m[n \mapsto v]\rangle$$

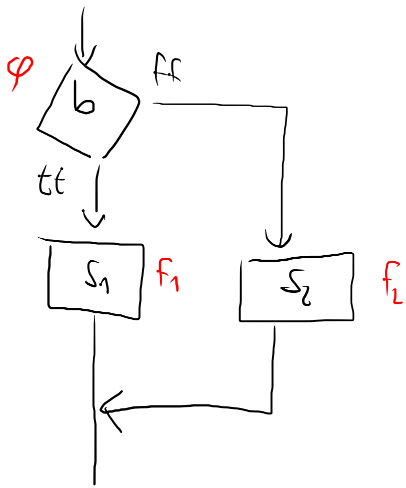
in memory $\quad$ at the position $\quad$ new value

	x	y	z	t
	0	1	2	3
m	10	14	7	20
		15		

$$\langle PUT\text{-}1 : c, 15, m\rangle \Rightarrow\!\!\!\!\!\!\Rightarrow \langle c, \varepsilon, m[1 \mapsto 15]\rangle$$

DENOTATIONAL SEMANTICS

if b then S_1 else S_2



$$f_1 = \mathcal{S}[\![S_1]\!]$$

$$f_2 = \mathcal{S}[\![S_2]\!]$$

$$\varphi = \mathcal{B}[\![b]\!]$$

$$\mathrm{cond}(\varphi, f_1, f_2)\, s = \begin{cases} f_1(s) & \varphi = \underline{tt} \\ f_2(s) & \varphi = \underline{ff} \end{cases}$$

Example

$$f : \mathbb{N} \to \mathbb{N} \qquad f(n) = \begin{cases} 1 & n = 0 \\ f(n+1) & n > 0 \end{cases}$$

$f(0) = 1$

$f(1) = f(2) = f(3) = \ldots \quad = 2$

$= 14$

functional $\quad F : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$

$$F^0 \bot (n) = id\, \bot(n) = id\, \bot = \bot \qquad \emptyset$$

$$F^1 \bot (n) = \begin{cases} 1 & n = 0 \\ F(F^0 \bot (n+1)) = \bot & n > 0 \end{cases} \qquad \mathrm{graph}(F^1 \bot) = \{(0,1)\}$$

$$F^2 \bot (n) = \begin{cases} 1 & n = 0 \\ F(F^1 \bot (n+1)) = \begin{cases} 1 & n+1 = 0 \\ & n = -1 \notin \mathbb{N} \\ F(F^0 \bot (n+2)) = \bot & \end{cases} & \end{cases}$$

$$\mathrm{graph}(F^2 \bot) = \{(0,1)\}$$

$\vdots$

$$(\mathrm{fix}\, F)(n) = \begin{cases} 1 & n = 0 \\ \bot & \text{oth.} \end{cases}$$

$$(\mathrm{fix}\, F)(n) = \lim_{n \to \infty} F^n \bot (n)$$

$(D, \sqsubseteq)$

let ω_1 the least elements
ω_2

$\omega_1 \sqsubseteq d$
$\omega_2 \sqsubseteq d$

$\omega_1 \sqsubseteq \omega_2$
$\omega_2 \sqsubseteq \omega_1$

(A) $\omega_1 = \omega_2$

$n | m$ $\quad n \sqsubseteq m$ $\quad D_{12} = \{1, 2, 3, 4, 6, 12\}$

$2 \sqsubseteq 12$ $\quad 12 \not\sqsubseteq 2$

$3 \sqsubseteq 12$ $\quad 2 \not\sqsubseteq 3$

not linear ordering

Ordering $\quad g, g'$ $\qquad gs = s' \Rightarrow g's = g's'$

$$g \sqsubseteq g'$$

$$f^3 s = f(f(f(s)))$$

POSET $\qquad (\underline{state} \to \underline{state}, \sqsubseteq)$

$$\omega \sqsubseteq f$$
$$\omega = \bot : \underline{State} \longrightarrow \emptyset$$
$$\text{graph}(\bot) = \emptyset$$

$\omega \sqsubseteq f \qquad \emptyset \subseteq \text{graph}(f)$

$$S = \text{if } (y \le x \wedge z \le x) \text{ then } max := x$$
$$\text{else if } (x \le y \wedge z \le y) \text{ then } max := y$$
$$\text{else } max := z;$$

$$S_{ds}[\![S]\!]s = \mathrm{cond}\big(B[\![\overbrace{y \le x \wedge z \le x}^{b_1}]\!],\ S_{ds}[\![max := x]\!],$$
$$f_2 = \mathrm{cond}(B[\![\underbrace{x \le y \wedge z \le y}_{b_2}]\!],\ S_{ds}[\![max := y]\!],\ S_{ds}[\![max := z]\!])\big)\, s$$

a) $s_0 = [x \mapsto \underline{10},\ y \mapsto \underline{14},\ z \mapsto \underline{34}]$

$$S_{ds}[\![S]\!]s_0 = \mathrm{cond}(B[\![b_1]\!],\ S_{ds}[\![max := x]\!],\ f_2)\, s_0 =$$
$$= f_2(s_0) = \mathrm{cond}(B[\![b_2]\!],\ \ldots,\ \ldots)\, s_0 =$$
$$= S_{ds}[\![max := z]\!]_{s_0} = s_0[max \mapsto \underline{34}]$$

b) $s_0' = [x \mapsto \underline{50},\ y \mapsto \underline{40},\ z \mapsto \underline{30}]$

```
a := x - y;
b := y - z;
c := x - z;
if (0 < a * b) then mid := y
               else if (0 < a * c) then mid := z
                                   else mid := x;
```

FIND A DENOTATION!

DENOTATION OF THE LOOP

$$S = \text{while } \neg(x=1) \text{ do } x := x-1$$

$$\mathcal{S}_{ds}[\![S]\!]s = (\text{fix } F)\, s$$

$$(F g)\, s = \text{cond}(\mathcal{B}[\![\neg(x=1)]\!],\ g \circ \mathcal{S}_{ds}[\![x := x-1]\!],\ \mathcal{S}_{ds}[\![\text{skip}]\!])\, s$$

$$= \text{cond}(\mathcal{B}[\![\neg(x=1)]\!],\ g(s[x \mapsto s\,x - 1]),\ \text{id})\, s$$

$$= \begin{cases} s & s\,x = \underline{1} \\ s[x \mapsto s\,x - \underline{1}] & s\,x \neq \underline{1} \end{cases}$$

$$F^0 \bot\, s = \bot$$

$$F^1 \bot\, s = \begin{cases} s & s\,x = \underline{1} \\ F(F^0 \bot\, s) = \bot & s\,x \neq \underline{1} \end{cases}$$

$$F^2 \bot\, s = \begin{cases} s & s\,x = \underline{1} \\ F(F^1 \bot\, s) = \begin{cases} s[x \mapsto s\,x - \underline{1}] & s\,x = \underline{2} \\ \bot & \text{oth.} \end{cases} \end{cases}$$

$$F^3 \bot\, s = \begin{cases} s & s\,x = \underline{1} \\ s[x \mapsto s\,x - \underline{1}] & s\,x = \underline{2} \\ s[x \mapsto s\,x \overset{-1-1}{-2}] & s\,x = \underline{3} \\ \bot & \text{oth.} \end{cases}$$

$\vdots$

$$F^n \bot\, s = \begin{cases} s & s\,x = \underline{1} \\ s[x \mapsto s\,x - (n-1)] & s\,x = n \\ \bot & \text{oth.} \end{cases}$$

$$(\text{fix } F)\, s = \begin{cases} s & s\,x = \underline{1} \\ s[x \mapsto \underline{1}] & s\,x > \underline{1} \\ \bot & s\,x < \underline{1} \end{cases}$$

$z := 0;$ while $(y \leq x)$ do $(z := z + 1;\ x := x - y)$

$y := 0;$ while $(1 \leq x)$ do $(y := y + 2 * x;\ x := x - 1)$