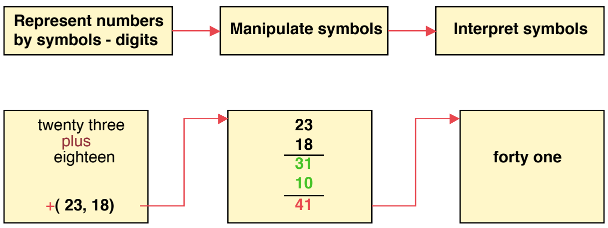
Alonzo Church
1903 - 1995

Model of Computation
based on λ–calculus

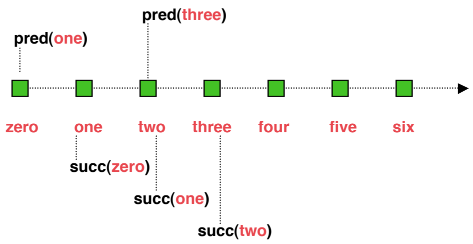


reason

manipulation of symbols can be mechanized - it does not require thinking

NN - natural numbers [0, 1, 2, …)

1. Zero recognition
2. Every ▢ has <u>one and only one</u> successor
3. Every ▢ except zero has <u>one and only one</u> predecessor



+ (x, y) ::= x | $_{y=o}$, + (succ (x), pred (y))

+		
+	5	7
	6	6
	7	5
	8	4
	9	3
	10	2
	11	1
	12	0

- (x, y) ::= x | $_{y=o}$, - (pred (x), pred (y))

-	5	2
	4	1
	3	0

* (x, y) ::= 0 | $_{y=o}$, + (x, * (x, pred (y)))

< (x, y) <u>iff</u> ∃ k ε NN: + (x, k) = y

< (3, 5) since 2 ε NN and + (3, 2) = 5

NN - representation & interpretation

SYNTAX

numeral ::= digit | numeral digit
digit ::= { 0, 1, 2, 3, …, r-1}

SEMANTICS

0 → zero
1 → one
2 → two
…

single digit numerals

… $d_3 d_2 d_1 d_0$

$d_0 * r^0$
$+ \; d_1 * r^1$
$+ \; d_2 * r^2$
$+ \; d_3 * r^3$
:

multi digit numerals

the meaning of the *composite numeral* is inferred
from the meaning of its *constituent parts*

meaning is a function that maps a string of d's into a unique number

OPERATIONS

+	4	2
	5	1
	6	0

Great, but what about + 13137 29251 ?

Answer Part 1: automate

12137 = 1*10⁴ + 3*10³ + 1*10² + 3*10¹ + 7*10⁰

$12137 = 1*10^4 + 3*10^3 + 1*10^2 + 3*10^1 + 7*10^0$

$29251 = 2*10^4 + 9*10^3 + 2*10^2 + 5*10^1 + 1*10^0$

since **(a + b) + (x + y) = (a +x) + (b + y)** and **(ax + bx = (a +b)x**

1	3	1	3	7	
2	9	2	5	1	
3	2	3	8	8	*sum*
1	0	0	0	0	*carry*
4	2	3	8	8	

Answer Part 2: automate further

32	16	8	4	2	1		
	1	0	1	1	1		23
	0	1	0	1	1		11
	1	1	1	0	0	*sum*	
	0	0	1	1		*carry*	
	1	1	0	1	0		
0	0	1	0	0			
0	1	0	0	1	0		
0	1	0	0	0			
0	0	0	0	1	0		
1	0	0	0	0			
1	**0**	**0**	**0**	**1**	**0**		34
0	0	0	0	0			
32	16	8	4	2	1		

Answer Part 3: automate further still

x \ y	0	1	
0	0 (sum), 0 (carry)	1 (sum), 0 (carry)	*sum* / *carry*
1	1 (sum), 0 (carry)	0 (sum), 1 (carry)	*sum* / *carry*

sum	**0**	**1**
0	0	1
1	1	0

↑ **x_or**

carry	**0**	**1**
0	0	0
1	0	1

↑ **and**

	1	**0**	**1**	**1**	**1**	a
	0	**1**	**0**	**1**	**1**	b
	1	1	1	0	0	a := a **x_or** b
	0	0	1	1		b := **l_shift** (a **and** b)
	1	1	0	1	0	⋮
0	0	1	0	0		
0	1	0	0	1	0	
0	1	0	0	0		
0	0	0	0	1	0	
1	0	0	0	0		
1	**0**	**0**	**0**	**1**	**0**	
0	0	0	0	0		

Integers

on Cartesian Product **NN** × **NN** = {(m, n): m, n ε **NN**)

define a relation ≈ $(m_1, n_1) \approx (m_2, n_2)$ iff $(m_1 + n_2) = (m_2 + n_1)$

≈ is relation of equivalence since it is reflexive, symmetric and transitive

Reflexive (m, n) ≈ (n, m) since m + n = n + m

Symmetric

if $(m_1, n_1) \approx (m_2, n_2)$ then $m_1 + n_2 = m_2 + n_1$ *by definition*

and $m_2 + n_1 = m_1 + n_2$

hence $(m_2, n_2) \approx (m_1, n_1)$

Transitive

suppose we have $(m_1, n_1) \approx (m_2, n_2)$ and $(m_2, n_2) \approx (m_3, n_3)$

by definition

$m_1 + n_2 = m_2 + n_1$

$m_2 + n_3 = m_3 + n_2$

adding sides

$m_1 + \cancel{n_2} + \cancel{m_2} + n_3 = \cancel{m_2} + n_1 + m_3 + \cancel{n_2}$

hence $m_1 + n_3 = n_1 + m_3$

and so $(m_1, n_1) \approx (m_3, n_3)$

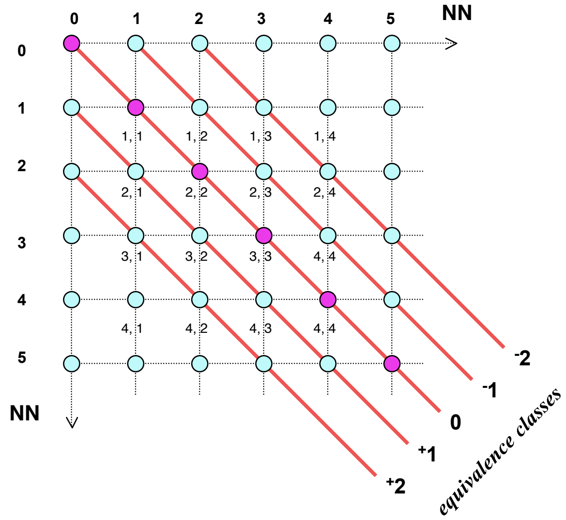
relation of equivalence in a non-empty set **X** divides this set into disjoint, non-empty subsets (classes of equivalence) in the following way:

two elements x, y ε **X** belong to the same class iff x ≈ y
|x| = {y ε **X** : x ≈ y}

integer may thus be defined as an equivalence class:

|(m, n)| = {(p, q) ε (**NN** × **NN**) **:** (m, n) ≈ (p, q)}

|(1, 1)|	= {(0, 0), (1, 1), (2, 2) …}	**integer zero**
|(1, 0)|	= {(1, 0), (2, 1), (3, 2) …}	**integer** $^{+}$**1**
|(0, 1)|	= {(0, 1), (1, 2), (2, 3) …}	**integer** $^{-}$**1**



Integer addition

$|(m_1, n_1)| \oplus |(m_2, n_2)| = |(m_1 + m_2), (n_1 + n_2)|$

Integer multiplication

$|(m_1, n_1)| \otimes |(m_2, n_2)| = |\ (m_1 \times m_2 + n_1 \times n_2)\ ,\ (m_1 \times n_2 + n_1 \times m_2)\ |$

NN-arithmetic is isomorphic to **IN-arithmetics**

Rational numbers can be defined similarly where

relation $\approx$ $(p_1, q_1) \approx (p_2, q_2)$ iff $(p_1 \otimes q_2) = (p_2 \otimes q_1)$
$p, q \in$ **IN**

It took us around 7000 years

We can

- **represent**
- **reason about**
- **process**

the **numbers** through **numerals i.e. in detachment from their meaning**

Can we do the same with even more abstract symbols ?

Suppose we need to evaluate the expression

(7 + x) * (8 + 5 * x) for **x = 4**

→ (7 + 4) * (8 + 5 * 4)
→ (7 + 4) * (8 + 20)
→ (7 + 4) * 28
→ 11 * 28
→ 308

→ (7 + 4) * (8 + 5 * 4)
→ 11 * (8 + 5 * 4)
→ 11 * (8 + 20)
→ 11 * 28
→ 308

Church-Rosser property - the order of evaluations is immaterial

evaluation may be applied to non-numerical symbols

first (sort (append (BANANA, LEMMON) (sort (GRAPE, APPLE, KIWI))))

first (sort (append (BANANA, LEMMON) (APPLE, GRAPE, KIWI)))

first (sort (BANANA, LEMMON, APPLE, GRAPE, KIWI))

first (APPLE , BANANA, GRAPE, LEMMON, KIWI)

APPLE

Do variable names matter?

$\int x\, dx \longleftrightarrow \int y\, dy$

not (A or B) ≡ (not A) and (not B) ⟷ not (X or Y) ≡ (not X) and (not Y)

but

$\int x \sin y\, dx \;\not\longleftrightarrow\; \int y \sin y\, dy$

What are the rules for?

$a(b + c)^2 - (ab^2 + ac^2 + abc)$ — *complex expression*

$\rightarrow a(b^2 + 2bc + c^2) - (ab^2 + ac^2 + abc)$

$\rightarrow (ab^2 + 2abc + ac^2) - (ab^2 + ac^2 + abc)$

$\rightarrow ab^2 + 2abc + ac^2 - ab^2 - ac^2 - abc$

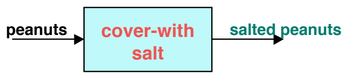
$\rightarrow 2abc - abc$

$\rightarrow abc$ — *simple(r) expression* f(x) = x + 5 **+** (x, 5) or **plus** (x, 5)

the meaning (function abstraction)

lambda expression → **f = λx. x + 5**

λx. salt-cover (x)



λx. salt-cover (x) (peanuts) → salt-cover_peanuts

λx. salt-cover (x) (meat) → salt-cover_meat

λx. salt-cover (x) (banana) → salt-cover_banana

λy. (λx. y-cover (x)) (sugar) → λx. sugar-cover (x)

λz. (λy. (λx. y-z (x))) (cover) → λy. (λx. y-cover (x))

λz. (λy. (λx. y-z (x))) (free) → λy. (λx. y-free (x))

(Haskell Curry)

currying

functions of n arguments can be represented by n-fold iteration of application

instead of applying the function to two arguments	**f(X, Y)**	**plus (3, 5)**
apply it to the first argument and then apply the result to the second argument	**(f(X))Y**	**((plus3) 5)**

more formally **(λ.(xy) F = λx. λy. F**

λ-expression ::= constant
| variable
| <λ-expression> < λ-expression > *application*
| λ-expression. < λ-expression > *abstraction*
| (λ-expression)

Notation

complex λ-expression	**M, N, P, Q, ...**	
variables	**x, y, z, ...**	
constants	**300000**	
application	**+5, add (5)** **+ 2 3 ≡ ((+2) 3)**	
MNPQ means	**(((MN)P))**	*(association to the left)*
built-in functions	e.g. **add ,** *neither constants nor λ-functions, defined for convenience, can be evaluated*	
abstraction	**λx. + 1 x** **(λx. (λy. * 5 y) (+ x 3))12**	

body

λx. x y

free — variable — must know its value (from outside)

bound — variable — argument of the function

variable declaration

+x (λx. + x 3) 4

free bound

(λx. xy (λy. y))

(λx. λ y. z ((λz. z (λx. y)))

free variables

(λx. λ y. xz (y z)) (λx. y (λy. y)))

MANIPULATING EXPRESSIONS

λx. + x 1 $\rightarrow_\alpha$ λy. + y 1

variable names are arbitrary

λx. (λ y. yx) $\not\rightarrow_\alpha$ λx. (λ x. xx)

but

λx. (λ y. yx) $\rightarrow_\alpha$ λx. (λ z. zx)

E

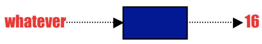
α -conversion rule

λx. E $\rightarrow_\alpha$ λz. [z $\leftarrow$ x] E

replace any bound **x** by **z** in **E**
provided that **z** doesn't occur in **E**

- λy. x $\not\rightarrow_\alpha$ λy. y — **x** is free in
- λx. (λ y. + x y) $\not\rightarrow_\alpha$ λx. (λ x. + xx) — **x** is free in
- λx. f x $\not\rightarrow_\alpha$ λx. g y — **f** is free in
- λx. (λ y. x y) $\rightarrow_\alpha$ λf. (λ y. f y)

- $(\lambda x.\ + x\ 5)\ 4 \to_\beta + 4\ 5$
- $(\lambda x.\ * x\ x)\ 5 \to_\beta * 5\ 5$
- $(\lambda x.\ 16)\ y \to_\beta 16$



- $(\lambda x.\ (\lambda y.\ * x\ y))\ 4\ 5 \to_\beta (\lambda y.\ * 4\ y)\ 5 \to_\beta * 4\ 5$
- $(\lambda a.\ a\ 2)\ (\lambda b.\ + b 1) \to_\beta (\lambda b.\ + b\ 1)\ 2 \to_\beta + 2\ 1$

β -conversion rule $(\lambda x.\ P)\ Q \to_\beta [\, x \leftarrow Q]\ P$ provided that bound variables of P are distinct from free variables of Q

all (free in P) x's in P get replaced by Q

- $(\lambda x.\ (\lambda y.\ x\ y))\ y \to_\beta \lambda y.\ y\ y$?! WRONG

bound free

$(\lambda x.\ (\lambda y.\ x\ y)\ y \to_\alpha (\lambda x.\ (\lambda z.\ x\ z)\ y$

$\to_\beta (\lambda z.\ y\ z)$

$\to_\alpha \lambda x.\ yx$

- $\lambda x.\ (\lambda\ y.\ \text{div}\ x\ y)\ 6\ 3$
  $\rightarrow_\beta (\lambda y.\ \text{div}\ 6\ y)\ 3$
  $\rightarrow_\beta \text{div}\ 6\ 3$
  $\rightarrow_\delta 2$

δ -conversion rule

evaluation of the built-in functions

- $\lambda x.\lambda\ y.\ +\ x\ ((\lambda x.\ -\ x\ 4)\ y)\ 5\ 6$
  $\rightarrow_\beta \lambda x.\ \lambda y.\ +\ x\ (\ -\ y\ 4)\ 5\ 6$
  $\rightarrow_\beta \lambda x.\ \ +\ x\ (\ -\ 5\ 4)\ 6$
  $\rightarrow_\beta +\ 6\ (\ -\ 5\ 4)$
  $^*\!\!\rightarrow_\delta +\ 7$

multiple application of δ-conversion

- $(\lambda x.\ +\ 5\ x)\ 4 \rightarrow_\beta +\ 5\ \ 4\ \rightarrow_\delta 9$

  $(\lambda x.\ +\ 5\ x)\ \rightarrow_\eta +\ 5$

  $(\lambda x.\ F\ x)\ \rightarrow_\eta F$

η -conversion rule

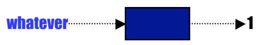
provided x does not occur free in F

λ-expression that contains no reducible sub-expression is said to be in normal form

- not every expression has a normal form, for instance
  (λx. x x) (λx. x x) → (λx. x x) (λx. x x) → (λx. x x) (λx. x x) → ...

- some reduction orders are more efficient than others:

(1) (λx. 1) (λx. x x) (λx. x x)
(λx. 1) (whatever) → 1



but

(2) (λx. 1)(λx. x x) (λx. x x) → (λx. 1)(λx. x x) (λx. x x) →...

NORMAL ORDER	APPLICATIVE ORDER
(λy. (λ x. (λz. (+ z x)) 4) y) 5	λy. (λ x. (λz. (+ z x)) 4) y) 5
→(λ x. (λz. (+ z x)) 4) 5	→ λy. (λ x. (+ 4 x) y) 5
→(λz. (+ z 5)) 4	→ λy. (+ 4 y) 5
→(+ 4 5)	→ (+ 4 5)
→ 9	→ 9

Church-Rosser Theorem

If $\lambda\text{-exp}_1 \leftrightarrow \lambda\text{-exp}_2$ then there exists $\lambda\text{-exp}$ such that

$\lambda\text{-exp}_1 \leftrightarrow \lambda\text{-exp}$

$\lambda\text{-exp}_2 \leftrightarrow \lambda\text{-exp}$

If $\lambda\text{-exp}_1 \leftrightarrow \lambda\text{-exp}_2$ and $\lambda\text{-exp}_2$ is in normal form then there exist a normal form reduction $\lambda\text{-exp}_1 \rightarrow \lambda\text{-exp}_2$

how does it work for numbers?

λf. λx. x	zero
λf. λx. f x	one
λf. λx. f (f x)	two
λf. λx. f (f (f x))	three
......	......

how many times **f** is applied to **x**

Church numerals

successor **succ ≡ λn.λf.λx.(f ((n f) x))**

succ zero ≡ λn.λf.λx. (f ((n f) x)) (λf.λx. x)

→ λf.λx. (f (λf.λx. x f) x)

→ λf.λx. (f (λg.λy. y g) x)

→ λf.λx. (f (λy. y) x)

→ λf.λx. (f x) → one

- **add ≡ λm.λn.λf.λx.(((m succ) n) f) x) f ((n f) x))**
  ≡ λm.λn.λf.λx.m f (n f x)

- **mult ≡ λm.λn.λf.m f (n f)**

- **exp ≡ λm.λn. (m n)**

... and for booleans?

- IDENTITY **λx. x**
- True **λx. λy. x**
- False **λx. λy. y**
- NOT **λx. (x F) T**
- AND **λx. λy ((x y)F)**
- OR **λx. λy ((x T) y)**

if .... then ... else?

- if **True B C → B** (1)
  if **False B C → C** (2)

  suppose we take **λx. x** for if

  then (1) becomes
  (λx. x) (λx. λy. x) B C → (λx. λy. x) B C → (λy. B) C → B

  and (2) becomes
  (λx. x) (λx. λy. y) B C → (λx. λy. y) B C → (λy. C) → C

... recursion?

- let
  Y≡ λf. (λx. f (x x)) ((λx. f (x x)) recursion combinator

  YR ≡ λf. (λx. f (x x)) ((λx. f (x x)) R

  → (λx. R (x x)) ((λx. R (x x))
  → R (λx. R (x x)) ((λx. R (x x))
  → ...

  keeps generating R's

  a function that calls (a function) **f** and regenerates itself

  YR ≡ R (YR)

λ-calculus

- processing functions by manipulating their abstractions using application and formal conversion rules
- everything in the computation process is represented by functions; there are no other objects or types (bool, int, chars, ...) ; if they are needed they must be represented via functions
- analysis of functions
  - without having to name them
  - seeing their abstractions at all times
  - being free from their intuitive properties

Church Thesis every intuitively computable function is λ-definable

- Turing machine
- μ-recursive functions (Gödel)
- λ-calculus (Church)
- formal grammars (Post)
- combinatory logic (Schönfinkel, Curry)

are computationally equivalent

normal order β-reduction models **lazy evaluation** for functional languages